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JHEP06(2009)082

(http://iopscience.iop.org/1126-6708/2009/06/082)

Published by IOP Publishing for SISSA

Received: *May 4, 2009*
Accepted: *June 2, 2009*
Published: *June 25, 2009*

JHEP06(2009)082

Quantum equivalence in Poisson-Lie T-duality

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Abstract: We prove that, general σ-models related by Poisson-Lie T-duality are quantum equivalent under one-loop renormalization group flow. We reveal general properties of the flows, we study the associated generalized coset models and provide explicit examples.

Keywords: Bosonic Strings, Renormalization Group, Sigma Models

ArXiv ePrint: 0904.4248

doi:10.1088/1126-6708/2009/06/082

Contents

JHEP06(2009)082

1 Introduction

A generalization of the well known Abelian [1] and non-Abelian [2] T-dualities is the so-called Poisson-Lie T-duality [3]. Its most notable feature is that it does not rely on the existence of isometries but rather on a rigid group-theoretical structure [3]. Nevertheless, it shares some common features with ordinary T-duality. For instance, it can be explicitly formulated as a canonical transformation between phase-space variables [4, 5], similarly to ordinary T-duality [6, 7], a property that seems to be very important for our considerations.[1]

An important question that was addressed successfully in a particular example in [10] is whether classically equivalent, via canonical transformations, models retain their equivalence beyond the classical level in the following sense. As two-dimensional field theories, these σ-models are renormalizable if the corresponding counter-terms, at a given order in a loop expansion, can be absorbed into a renormalization of the various coupling constants appearing in the model up to field redefinitions or, equivalently, diffeomorphisms in the target space. These give rise to beta-function renormalization group equations which generally form a non-linear coupled system of first order differential equations. The classical equivalence of the two models can be promoted order by order in perturbation theory into

[1] Related developments include works on open string boundary conditions (see, for instance, [8, 9]).

the quantum level if the two different renormalization group flow systems of equations are in fact equivalent. In turn, this strongly hints towards their equivalence beyond the classical level. The existence of a canonical transformation relating two different σ-models seems to be necessary for their equivalence at the quantum level.[2] A general proof for all Poisson-Lie T-duality related models requires the generalized Ricci tensor. The latter was computed in [13] where it was shown that Poisson-Lie T-duality related models are renormalizable in the above described sense.

The main objective of the present work is to show the equivalence of the renormalization group flow for general σ-models related by Poisson-Lie T-duality. We study properties of the flows and prove several general statements regarding mainly the possible truncations of the parameter space in a way consistent with the renormalization group equations. We also investigate the generalized coset models introduced in [14] at the purely classical level. Finally, we present some explicit examples based on three-dimensional algebras.

2 Brief review of Poisson-Lie T-duality

In the section we review the most relevant aspects of Poisson-Lie T-duality for our purposes by following the conventions of [4, 5, 10, 14]. In the absence of spectator fields, the dual two-dimensional σ-model actions are [3]

$$S = \frac{1}{2\lambda}\int E_{ab}L^a_\mu L^b_\nu \partial_+ X^\mu \partial_- X^\nu \,, \qquad E = (M-\Pi)^{-1}\,, \tag{2.1}$$

and

$$\tilde{S} = \frac{1}{2\lambda}\int \tilde{E}^{ab}\tilde{L}_{a\mu}\tilde{L}_{\beta\nu}\partial_+ \tilde{X}^\mu \partial_- \tilde{X}^\nu \,, \qquad \tilde{E} = (M^{-1}-\tilde{\Pi})^{-1}\,. \tag{2.2}$$

The field variables X^μ and $\tilde{X}^\mu$, with $\mu = 1, 2, \ldots, d_G$ parametrize elements g and $\tilde{g}$ of two groups G and $\tilde{G}$, respectively, of equal dimension. We introduce representation matrices, $\{T_a\}$ and $\{\tilde{T}^a\}$ with $a = 1, 2, \ldots, d_G$, of the associated Lie algebras which form a pair of maximally isotropic subalgebras into which the Lie algebra of a group, known as the Drinfeld double, can be decomposed.[3] The commutation relations are

$$\begin{aligned}
[T_a, T_b] &= i f_{ab}{}^c T_c \,, \\
\left[\tilde{T}^a, \tilde{T}^b\right] &= i \tilde{f}^{ab}{}_c \tilde{T}^c \,, \\
\left[T_a, \tilde{T}^b\right] &= i \tilde{f}^{bc}{}_a T_c - i f_{ac}{}^b \tilde{T}^c
\end{aligned} \tag{2.3}$$

and imply the Jacobi identities (in our conventions $(ab) = ab + ba$ and $[ab] = ab - ba$)

$$\begin{aligned}
f_{ab}{}^d f_{dc}{}^e + f_{ca}{}^d f_{db}{}^e + f_{bc}{}^d f_{da}{}^e &= 0\,, \\
\tilde{f}^{ab}{}_d \tilde{f}^{dc}{}_e + \tilde{f}^{ca}{}_d \tilde{f}^{db}{}_e + \tilde{f}^{bc}{}_d \tilde{f}^{da}{}_e &= 0\,, \\
f_{ab}{}^d \tilde{f}^{ce}{}_d + f_{d[a}{}^c \tilde{f}^{de}{}_{b]} - f_{d[a}{}^e \tilde{f}^{dc}{}_{b]} &= 0\,.
\end{aligned} \tag{2.4}$$

[2] As a counterexample note the case of the Principal Chiral model and the Pseudodual Chiral model [11] whose classical solutions are in one-to-one correspondence but which are not canonically equivalent [6]. It is well known that their quantum behaviors are drastically different [12].

[3] A generalization of Poisson-Lie T-duality named Poisson-Lie T-plurality [15]-[17] is based on the non-uniqueness of the decomposition of the Drinfeld double.

JHEP06(2009)082

Introducing a bilinear invariant $\langle\cdot|\cdot\rangle$ with the various representation matrices obeying

$$\langle T_a|T_b\rangle = \langle\tilde{T}^a|\tilde{T}^b\rangle = 0\,, \qquad \langle T_a|\tilde{T}^b\rangle = \delta_a{}^b\,, \tag{2.5}$$

we define the components of the Maurer-Cartan forms appearing in the above actions as

$$L^a_\mu = -i\langle g^{-1}\partial_\mu g|\tilde{T}^a\rangle\,, \qquad \tilde{L}_{a\mu} = -i\langle\tilde{g}^{-1}\partial_\mu\tilde{g}|T_a\rangle\ . \tag{2.6}$$

The overall coupling constant is λ in (2.1) and (2.2) is assumed to be positive and the square matrix M is constant and has dimension d_G. The matrices Π and $\tilde{\Pi}$ depend on the variables X^μ and $\tilde{X}^\mu$ via the corresponding group elements g and $\tilde{g}$, respectively. They are defined as

$$\Pi^{ab} = b^{ca}a_c{}^b\,, \qquad \tilde{\Pi}_{ab} = \tilde{b}_{ca}\tilde{a}^c{}_b\,, \tag{2.7}$$

where the matrices $a(g)$, $b(g)$ are constructed using

$$g^{-1}T_a g = a_a{}^b T_b\,, \qquad g^{-1}\tilde{T}^a g = b^{ab}T_b + (a^{-1})_b{}^a\tilde{T}^b\ . \tag{2.8}$$

Consistency restricts them to obey

$$a(g^{-1}) = a^{-1}(g)\,, \qquad b^T(g) = b(g^{-1})\,, \qquad \Pi^T(g) = -\Pi(g)\,, \tag{2.9}$$

We also define the dual tilded symbols with similar properties.

3 Renormalization group and Poisson-Lie T-duality

We begin this section with a short review of the renormalization group in two-dimensional field theories with curved target spaces. The σ-models (2.1) and (2.2) are of the form

$$S = \frac{1}{2\lambda}\int Q^+_{\mu\nu}\partial_+X^\mu\partial_-X^\nu\,, \qquad Q^+_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}\ . \tag{3.1}$$

As a two-dimensional field theory for the fields X^μ, this will be renormalizable if the corresponding counter-terms, at a given order in a loop expansion, can be absorbed into a renormalization of the coupling constant λ and (or) of some parameters labeled collectively by a^i, $i = 1, 2, \ldots$. In our case these are the entries of the matrix M^{ab} and the overall coupling constant λ. In doing so, we may allow for general field redefinitions of the X^μ's, which are coordinate reparametrizations in the target space. This definition of renormalizability of σ-models is quite strict and similar to that for ordinary field theories. An extension of this is to allow for the manifold to vary with the mass scale and the renormalization group to act in the infinite-dimensional space of all metrics and torsions [18], but this will not be needed for our purposes.

The one-loop beta-functions are expressed as

$$\frac{d\lambda}{dt} = \frac{\lambda^2}{\pi}J_1\,, \qquad \frac{da^i}{dt} = \frac{\lambda}{\pi}a^i_1\,, \tag{3.2}$$

where $t = \ln\mu$, with μ being the mass energy scale and J_1, a^i_1 and X^μ_1 are chosen so that

$$\frac{1}{2}R^-_{\mu\nu} = -J_1Q^+_{\mu\nu} + \partial_{a^i}Q^+_{\mu\nu}a^i_1 + \partial_\lambda Q^+_{\mu\nu}X^\lambda_1 + Q^+_{\lambda\nu}\partial_\mu X^\lambda_1 + Q^+_{\mu\lambda}\partial_\nu X^\lambda_1\ . \tag{3.3}$$

Here $R^-_{\mu\nu}$ are the components of the generalized Ricci tensor defined with a connection that includes the torsion, i.e. $\Gamma^\mu_{\nu\rho} - \frac{1}{2}H^\mu{}_{\nu\rho}$. The counter-terms were computed in [18–20].

3.1 One loop renormalization group for Poisson-Lie T-duals

For the Poisson-Lie T-duality related actions (2.1) and (2.2) it was first shown that it is possible to satisfy the system (3.3) in the case of a six-dimensional Drinfeld double in [10, 14].[4] This made plausible that it could be true for at least a large class of such doubles. A general proof requires the computation of the generalized Ricci tensor for each σ-model separately. Using the underlying Poisson-Lie structure and in particular some useful identities derived in [5] this was done in [13],[5] where it was also shown that the models are one-loop renormalizable in the sense explained above.

We define for convenience the following quantities

$$A^{ab}{}_c = \tilde{f}^{ab}{}_c - f_{cd}{}^a M^{db}\,, \qquad B^{ab}{}_c = \tilde{f}^{ab}{}_c + M^{ad} f_{dc}{}^b\,, \tag{3.4}$$

as well as their duals

$$\tilde{A}_{ab}{}^c = f_{ab}{}^c - \tilde{f}^{cd}{}_a (M^{-1})_{db}\,, \qquad \tilde{B}_{ab}{}^c = f_{ab}{}^c + (M^{-1})_{ad} \tilde{f}^{dc}{}_b\,, \tag{3.5}$$

Using these we construct also

$$\begin{aligned} L^{ab}{}_c &= \frac{1}{2}(M_s^{-1})_{cd}\left(B^{ab}{}_e M^{ed} + A^{db}{}_e M^{ae} - A^{ad}{}_e M^{eb}\right)\,, \\ R^{ab}{}_c &= \frac{1}{2}(M_s^{-1})_{cd}\left(A^{ab}{}_e M^{de} + B^{ad}{}_e M^{eb} - B^{db}{}_e M^{ae}\right) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \tilde{L}_{ab}{}^c &= \frac{1}{2}(\tilde{M}_s^{-1})^{cd}\left(\tilde{B}_{ab}{}^e (M^{-1})_{ed} + \tilde{A}_{db}{}^e (M^{-1})_{ae} - \tilde{A}_{ad}{}^e (M^{-1})_{eb}\right)\,, \\ \tilde{R}_{ab}{}^c &= \frac{1}{2}(\tilde{M}_s^{-1})^{cd}\left(\tilde{A}_{ab}{}^e (M^{-1})_{de} + \tilde{B}_{ad}{}^e (M^{-1})_{eb} - \tilde{B}_{db}{}^e (M^{-1})_{ae}\right)\,, \end{aligned} \tag{3.7}$$

where[6]

$$M_s = \frac{1}{2}\left(M + M^T\right)\,, \qquad \tilde{M}_s = \frac{1}{2}\left[(M^{-1}) + (M^{-1})^T\right]\,. \tag{3.10}$$

From the results in [13] we deduce that the one-loop renormalization group flow system of equations corresponding to (2.1) is

$$\frac{dM^{ab}}{dt} = \frac{\lambda}{2\pi} R^{ac}{}_d L^{db}{}_c\,. \tag{3.11}$$

[4]For Abelian and non-Abelian dualities, similar investigation were performed for a one-parameter family deformations of the Principal Chiral model for SU(2), and its non-Abelian dual [6, 21], in [21].

[5]To compare our conventions with those in [13] one should perform the following replacements in that paper. Namely, we send $g \to g^{-1}$, $\Pi \to -\Pi$ and $\tilde{\Pi} \to -\tilde{\Pi}$.

[6]Note also the identities

$$R^{ab}{}_b = \tilde{f}^{ab}{}_b + M^{ab} f_{bc}{}^c + (M M_s^{-1} M)^{bc} f_{bc}{}^a\,, \qquad R^{ba}{}_b = 0 \tag{3.8}$$

and

$$L^{ab}{}_b = 0\,, \qquad L^{ba}{}_b = -\tilde{f}^{ab}{}_b + M^{ba} f_{bc}{}^c - (M M_s^{-1} M)^{bc} f_{bc}{}^a\,. \tag{3.9}$$

Similar relations hold for the tilded dual quantities.

JHEP06(2009)082

Similarly, for its dual (2.2) we have

$$\frac{d(M^{-1})_{ab}}{dt} = \frac{\lambda}{2\pi}\tilde{R}_{ac}{}^{d}\tilde{L}_{db}{}^{c}\,, \tag{3.12}$$

Also it turns out that the overall coupling λ does not get renormalized, as noticed already in a particular example in [14]. Hence, we may, in the remaining of this paper absorb, for notational convenience, the overall factor $\frac{\lambda}{2\pi}$ into a redefinition of t.

3.2 Proof of compatibility

In order to show that Poisson-Lie T-duality holds at the one-loop quantum level we have to demonstrate compatibility with the renormalization flow equations. This will be explicit if the two systems (3.11) and (3.12) are if fact equivalent.

We first note the, easy to prove, relations

$$A^{ab}{}_{c} = -M^{db}\tilde{A}_{cd}{}^{a}\,, \qquad B^{ab}{}_{c} = M^{ad}\tilde{B}_{dc}{}^{b}\,, \tag{3.13}$$

their dual (which give no further information)

$$\tilde{A}_{ab}{}^{c} = -(M^{-1})_{db}A^{cd}{}_{a}\,, \qquad \tilde{B}_{ab}{}^{c} = (M^{-1})_{ad}B^{dc}{}_{b} \tag{3.14}$$

and that

$$M_s = M\tilde{M}_s M^T = M^T\tilde{M}_s M\ . \tag{3.15}$$

Next we note the useful identities

$$R^{ab}{}_{c} = M^{ad}M^{eb}(M^{-1})_{fc}\tilde{R}_{de}{}^{f}\,, \qquad L^{ab}{}_{c} = -M^{ad}M^{eb}(M^{-1})_{cf}\tilde{L}_{de}{}^{f}\ . \tag{3.16}$$

To prove them we start form the right hand side in the first identity which using (3.14) and (3.15) can be cast into the following form

$$\frac{1}{2}(M_s^{-1})_{cd}\left(-M^{ae}A^{db}{}_{e} + M^{de}B^{ab}{}_{e} - M^{eb}B^{da}{}_{e}\right)\ . \tag{3.17}$$

To proceed we note that from the definitions (3.4) we have

$$\begin{aligned} A^{db}{}_{e} &= B^{db}{}_{e} - M^{df}f_{fe}{}^{b} - M^{fb}f_{ef}{}^{d}\,,\\ B^{ab}{}_{e} &= A^{ab}{}_{e} + M^{af}f_{fe}{}^{b} + M^{fb}f_{ef}{}^{a}\,,\\ B^{da}{}_{e} &= -B^{ad}{}_{e} + M^{af}f_{fe}{}^{d} + M^{df}f_{fe}{}^{a}\ . \end{aligned} \tag{3.18}$$

Substituting into (3.17) we obtain

$$\begin{aligned} &\frac{1}{2}(M_s^{-1})_{cd}\left(A^{ab}{}_{e}M^{de} + B^{ad}{}_{e}M^{eb} - B^{db}{}_{e}M^{ae}\right)\\ &+\frac{1}{2}(M_s^{-1})_{cd}\Big(f_{fe}{}^{b}M^{ae}M^{df} + f_{ef}{}^{d}M^{ae}M^{fb}\\ &\qquad + f_{fe}{}^{b}M^{de}M^{af} + f_{ef}{}^{a}M^{de}M^{fb} - f_{fe}{}^{d}M^{eb}M^{af} - f_{fe}{}^{a}M^{eb}M^{df}\Big)\ . \end{aligned} \tag{3.19}$$

The parenthesis in the first line is simply $R^{ab}{}_c$, whereas the remaining terms vanish. The second identity in (3.16) follows from a similar computation. Alternatively, one can prove it by noticing the transformation

$$M \to -M^T \qquad \Longrightarrow \qquad R^{ab}{}_c \longleftrightarrow -L^{ba}{}_c\,, \quad \tilde{R}_{ab}{}^c \longleftrightarrow -\tilde{L}_{ba}{}^c\,. \tag{3.20}$$

Using (3.16) one easily sees that the system (3.11) implies (3.12) and vice versa.

Hence, we conclude that, at one-loop in perturbation theory, general σ-models related by Poisson-Lie T-duality are equivalent under the renormalization group flow.

4 Properties of renormalization group flow

In this section we study several properties of the renormalization group equations. First note that, setting the matric M proportional to the identity is not consistent with the renormalization group equations for general Drinfeld doubles. It is, however, consistent for abelian $\tilde{G}$'s, in the basis where the identity is the invariant metric for the group G.

Consider next a subgroup H of G and another $\tilde{H}$, of equal dimension, subgroup of $\tilde{G}$. We split the index $a = (i, \alpha)$, where $i = 1, 2, \ldots, \dim(H)$ and $\alpha = 1, 2, \ldots, \dim(G/H)$. We note in passing that, H and $\tilde{H}$ form a Drinfeld double as well. This can be easily seen by restricting the four free indices in the mixed Jacobi identity in (2.4) to the subgroup.

We will use the following notation for the matrices M^{ab} and Π^{ab}

$$(M^{ab}) = \begin{pmatrix} H^{ij} & M^{i\beta} \\ M^{\alpha j} & K^{\alpha\beta} \end{pmatrix}\,, \qquad (\Pi^{ab}) = \begin{pmatrix} \Pi_0^{ij} & \Pi_2^{i\alpha} \\ -\Pi_2^{j\beta} & \Pi_1^{\alpha\beta} \end{pmatrix}\,. \tag{4.1}$$

and similarly for $\tilde{\Pi}_{ab}$.

4.1 Consistent truncation of the parameter space

A natural question to investigate is to what an extend we may choose in the matrix (4.1) the mixed-index elements to be zero, namely that

$$M^{i\alpha} = M^{\alpha i} = 0\,. \tag{4.2}$$

We will find the conditions under which the matrix M remains of a block diagonal form, under the renormalization group flow. In general we have

$$\frac{dM^{i\alpha}}{dt} = R^{ic}{}_d L^{d\alpha}{}_c = R^{ij}{}_k L^{k\alpha}{}_j + R^{i\beta}{}_k L^{k\alpha}{}_\beta + R^{ij}{}_\beta L^{\beta\alpha}{}_j + R^{i\beta}{}_\gamma L^{\gamma\alpha}{}_\beta\,. \tag{4.3}$$

With the choice (4.2) and using that $f_{ij}{}^\alpha = f_{k\beta}{}^l = 0$, it is easy to compute that the quantities defined in (3.4)–(3.7) are zero when two indices are latin letters and one is a greek one. In addition, the form of $L^{\gamma\alpha}{}_\beta$ is that of (3.6) with the latin-letter indices replaced by greek ones, hence projected completely into the coset space. We conclude that, in the right hand side of (4.3) only the last term is non-vanishing. Hence, a sufficient condition to preserve the choice (4.2) under renormalization group flow is that the coset spaces G/H and $\tilde{G}/\tilde{H}$ are symmetric, i.e. the structure constants $f_{\alpha\beta}{}^\gamma = \tilde{f}^{\alpha\beta}{}_\gamma = 0$. This is not a

necessary condition as well, even in the abelian $\tilde{G}$ case. An example is the non-symmetric coset space SU(3)/SU(2). We will present the details in section 5.

Finally, for later use, we write the factor

$$R^{i\beta}{}_{\gamma} = \frac{1}{2}(K_s^{-1})_{\gamma\delta}\left(H^{il}(f_{l\zeta}{}^{\delta}K^{\zeta\beta} + f_{l\zeta}{}^{\beta}K^{\delta\zeta} - \tilde{f}^{\delta\beta}{}_{l}) + K^{\delta\zeta}(\tilde{f}^{i\beta}{}_{\zeta} - f_{\zeta\eta}{}^{i}K^{\eta\beta}) - \tilde{f}^{i\delta}{}_{\zeta}K^{\zeta\beta}\right) . \tag{4.4}$$

4.2 Generalized coset spaces

Let's examine the behaviour and equivalence under renormalization group flow of a class of σ-models introduced at the purely classical level in [14]. Consider the limit

$$H^{ij} \to \infty \quad \Longleftrightarrow \quad (H^{-1})_{ij} \to 0\,, \tag{4.5}$$

in a uniform way for all matrix elements, meaning that ratios of matrix elements remain constant in this limit. Then, the action (2.1) takes the form

$$S = \frac{1}{2\lambda}\int \Sigma_{\alpha\beta}L_{\mu}^{\alpha}L_{\nu}^{\beta}\partial_{+}X^{\mu}\partial_{-}X^{\nu}\ . \qquad \Sigma = (K - \Pi_1)^{-1}\ . \tag{4.6}$$

Since

$$M^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & K^{-1} \end{pmatrix} + \mathcal{O}\left(H^{-1}\right)\,, \tag{4.7}$$

the dual action (2.2) becomes

$$\tilde{S} = \frac{1}{2\lambda}\int \tilde{\Sigma}^{AB}\tilde{L}_{A\mu}\tilde{L}_{B\nu}\partial_{+}\tilde{X}^{\mu}\partial_{-}\tilde{X}^{\nu}\,, \qquad \tilde{\Sigma} = \begin{pmatrix} \tilde{\Pi}_0 & \tilde{\Pi}_2 \\ -\tilde{\Pi}_2 & K^{-1} - \tilde{\Pi}_1 \end{pmatrix}\ . \tag{4.8}$$

Upon taking the limit (4.5) the number of variables in the two actions has been reduced to $\dim(G/H)$. The reduced dimensionality of (2.1) happens since, after taking the limit (4.5), a local gauge invariance develops provided that certain conditions hold. In particular, it has been shown that (4.6) is invariant under the local gauge transformation $g \to gh$, with $h \in H$, provided that the following conditions hold [14]

$$\tilde{f}^{\alpha\beta}{}_{i} = f_{i\gamma}{}^{\alpha}K^{\gamma\beta} + f_{i\gamma}{}^{\beta}K^{\alpha\gamma}\ . \tag{4.9}$$

Note that for abelian group $\tilde{G}$ this reduces to the usual condition in cosets G/H for an invariant tensor [22] so that (4.9) presents the analog of this condition for our generalized cosets. Also, when $\tilde{G}$ is non-abelian it is not consistent to take K to be a symmetric matrix. Then, we may gauge-fix the $\dim(H)$ parameters in the group element $g \in G$. The most efficient way, that completely fixes the gauge, is to parametrize as $g = \kappa h$, where $h \in H$ and $\kappa \in G/H$, and then set $h = I$.

4.2.1 Renormalization group flow

We would like first to investigate if the limit (4.5) is consistent with the renormalization group equations (3.11), (3.12). In general

$$\frac{dK^{-1}_{\alpha\beta}}{dt} = \tilde{R}_{\alpha c}{}^{d}\tilde{L}_{d\beta}{}^{c} = \tilde{R}_{\alpha i}{}^{j}\tilde{L}_{j\beta}{}^{i} + \tilde{R}_{\alpha i}{}^{\gamma}\tilde{L}_{\gamma\beta}{}^{i} + \tilde{R}_{\alpha\gamma}{}^{i}\tilde{L}_{i\beta}{}^{\gamma} + \tilde{R}_{\alpha\gamma}{}^{\delta}\tilde{L}_{\delta\beta}{}^{\gamma}\ . \tag{4.10}$$

JHEP06(2009)082

In the limit (4.5), $(M^{-1})_{i\alpha}$ and $(M^{-1})_{\alpha i}$ as well as the quantities (3.4)–(3.7) when two of the indices are latin letters and one is a greek one, are of order $(H^{-1})_{ij}$. Hence, the first term in (4.10) vanishes. Among the remaining terms the last one is actually independent of the matrix H. In the second term the factor $\tilde{R}_{\alpha i}{}^{\gamma}$ has a finite part and a vanishing part under the limit (4.5). Similarly, the second factor $\tilde{L}_{\gamma\beta}{}^{i}$ has a divergent part and a finite part under this limit. The divergent part is

$$\frac{1}{2}(\tilde{H}_s^{-1})^{ij}\left[(f_{j\beta}{}^{\delta}-\tilde{f}^{\delta\epsilon}{}_{j}K^{-1}_{\epsilon\beta})K^{-1}_{\gamma\delta}-f_{\gamma j}{}^{\delta}K^{-1}_{\delta\beta}\right]\ . \tag{4.11}$$

It can be easily seen that it vanishes upon using (4.9). Therefore, to compute in the limit (4.5), the term $\tilde{R}_{\alpha i}{}^{\gamma}\tilde{L}_{\ g\beta}{}^{i}$ we need to keep the finite part of both factors. With some rearrangement we obtain

$$\frac{1}{4}(HH_s^{-1})^{i}{}_{k}\left[f_{\alpha i}{}^{\epsilon}K^{-1}_{\delta\epsilon}-K^{-1}_{\alpha\epsilon}(f_{\delta i}{}^{\epsilon}+K^{-1}_{\delta\gamma}\tilde{f}^{\gamma\epsilon}{}_{i})\right]\left[f_{\gamma\beta}{}^{k}+K^{-1}_{\gamma\zeta}\tilde{f}^{\zeta k}{}_{\beta}+K^{-1}_{\zeta\beta}\tilde{f}^{\zeta k}{}_{\gamma}\right]\ . \tag{4.12}$$

Using (4.9) we finally obtain

$$(HH_s^{-1})^{i}{}_{k}\Gamma_{i\alpha\beta}{}^{k}\,, \tag{4.13}$$

where

$$\Gamma_{i\alpha\beta}{}^{k}=\frac{1}{2}f_{i\eta}{}^{\delta}K^{\eta\gamma}(K^{-1})_{\alpha\delta}(f_{\gamma\beta}{}^{k}+K^{-1}_{\gamma\zeta}\tilde{f}^{\zeta k}{}_{\beta}+K^{-1}_{\zeta\beta}\tilde{f}^{\zeta k}{}_{\gamma})\ . \tag{4.14}$$

To this term we should add a similar one coming from the third term in (4.10) which using the transformation (3.20) reads

$$(H^{T}H_s^{-1})^{i}{}_{k}\Delta_{i\alpha\beta}{}^{k}\,, \tag{4.15}$$

where

$$\Delta_{i\alpha\beta}{}^{k}=\frac{1}{2}f_{i\eta}{}^{\delta}K^{\gamma\eta}(K^{-1})_{\delta\beta}(f_{\gamma\alpha}{}^{k}-K^{-1}_{\zeta\gamma}\tilde{f}^{\zeta k}{}_{\alpha}-K^{-1}_{\alpha\zeta}\tilde{f}^{\zeta k}{}_{\gamma})\ . \tag{4.16}$$

It is clear that, in order for the limit (4.5) to be independent of the form of the matrix H the latter has to be symmetric.[7] Denoting by

$$\Gamma_{\alpha\beta}=\Gamma_{i\alpha\beta}{}^{i}\,,\qquad \Delta_{\alpha\beta}=\Delta_{i\alpha\beta}{}^{i}\,, \tag{4.17}$$

we finally have that, in the limit (4.5)

$$\frac{dK^{-1}_{\alpha\beta}}{dt}=\Gamma_{\alpha\beta}+\Delta_{\alpha\beta}+\tilde{R}_{\alpha\gamma}{}^{\delta}\tilde{L}_{\delta\beta}{}^{\gamma}\ . \tag{4.18}$$

This would have been the result for the renormalization group flow system of equations for the dual coset model (4.8). Note that for symmetric spaces the last term is zero and the running is solely due to the first two terms. These could be thought of as a remnant of the original full group structure in (2.2). Had we performed a similar limiting procedure starting from the system (3.11) corresponding to the action (4.6) we would have obtained

$$\frac{dK^{\alpha\beta}}{dt}=P^{\alpha\beta}+Q^{\alpha\beta}+R^{\alpha\gamma}{}_{\delta}L^{\delta\beta}{}_{\gamma}\,, \tag{4.19}$$

[7] Alternatively, we may slightly modify the limit (4.5) to involve only the symmetric part of H.

where

$$P^{\alpha\beta} = P_i{}^{\alpha\beta i}\,, \qquad Q^{\alpha\beta} = Q_i{}^{\alpha\beta i}\,, \tag{4.20}$$

with

$$\begin{aligned} P_i{}^{\alpha\beta k} &= \frac{1}{2} f_{i\gamma}{}^{\alpha}\left[K^{\gamma\eta}(\tilde{f}^{k\beta}{}_{\eta} - f_{\eta\zeta}{}^{k}K^{\zeta\beta}) + \tilde{f}^{k\gamma}{}_{\zeta}K^{\zeta\beta}\right]\,, \\ Q_i{}^{\alpha\beta k} &= -\frac{1}{2} f_{i\gamma}{}^{\beta}\left[K^{\eta\gamma}(\tilde{f}^{k\alpha}{}_{\eta} + f_{\eta\zeta}{}^{k}K^{\alpha\zeta}) + \tilde{f}^{k\gamma}{}_{\zeta}K^{\alpha\zeta}\right]\,. \end{aligned} \tag{4.21}$$

4.2.2 One loop equivalence

The equivalence of the systems (4.18) and (4.19) can be demonstrated by noting that there exist expressions similar to (3.16) projected completely to the coset space indices, namely that

$$R^{\alpha\beta}{}_{\gamma} = K^{\alpha\delta}K^{\eta\beta}(K^{-1})_{\zeta\gamma}\tilde{R}_{\delta\eta}{}^{\zeta}\,, \qquad L^{\alpha\beta}{}_{\gamma} = -K^{\alpha\delta}K^{\eta\beta}(K^{-1})_{\eta\zeta}\tilde{L}_{\delta\eta}{}^{\zeta} \tag{4.22}$$

and in addition one may prove that

$$P^{\alpha\beta} = -K^{\alpha\gamma}K^{\delta\beta}\Gamma_{\gamma\delta}\,, \qquad Q^{\alpha\beta} = -K^{\alpha\gamma}K^{\delta\beta}\Delta_{\gamma\delta}\,. \tag{4.23}$$

Next we investigate when the condition (4.9) for being able to take the limit (4.5) consistently, is preserved under the renormalization group flow. We demand that

$$f_{i\gamma}{}^{\alpha}\frac{dK^{\gamma\beta}}{dt} + f_{i\gamma}{}^{\beta}\frac{dK^{\alpha\gamma}}{dt} = 0\,. \tag{4.24}$$

The only possibility that this is obeyed is that the right hand side of (4.19) is an invariant tensor obeying a condition similar to (4.9). In the case with $\tilde{f}^{ab}{}_{c} = 0$, (4.24) can be proven by repeatedly using the Jacobi identity for the $f_{ab}{}^{c}$'s, the coset constraint on $K^{\alpha\beta}$ (4.9) and by taking $K_s^{\alpha\beta}$ to be block diagonal. This can always be done by a proper transformation of the group element $g \in G$ on the action.[8] For the case with $\tilde{f}^{ab}{}_{c} \neq 0$ the proof is much more complicated and we have just explicitly checked that this is indeed true for the cosets of subsection 5.1 below.

Finally, we note that the limit (4.5) is not a fixed point of the renormalization group flow in general. Consider the special case of $H^{ij} = H\delta^{ij}$ with δ^{ij} being the invariant metric of the group. Then after some computations we obtain that

$$\frac{dH^{-1}_{ij}}{dt} = \frac{1}{4}f_{ikl}f^{jkl} - \frac{1}{4}(K_s^{-1})_{\alpha\gamma}(K_s^{-1})_{\beta\delta}\tilde{f}^{\alpha\beta}{}_{i}\tilde{f}^{\gamma\delta}{}_{j} + \mathcal{O}\left(H^{-1}\right)\,. \tag{4.26}$$

Note that the matrix remains under renormalization group flow symmetric, but no longer diagonal. Also it is zero for abelian subgroups H, when simultaneously $\tilde{f}^{\alpha\beta}{}_{i} = 0$.

The off diagonal elements do also flow even in the coset limit unless the space is symmetric. Note that the flow is well defined since the coefficient of the H-dependent term in (4.4) vanishes thanks to the condition (4.9).

[8] In the proof we also use the fact that

$$f_{i\gamma}{}^{\eta}f_{c\eta}{}^{d}f_{d\zeta}{}^{c}\,, \qquad f_{i\gamma}{}^{\varepsilon}f_{\delta\varepsilon}{}^{\eta}f_{\eta\zeta}{}^{\delta}\,, \tag{4.25}$$

are antisymmetric under the interchange of γ and ζ.

5 Examples

5.1 Six-dimensional doubles

In this section we first construct several examples based on a six-dim Drinfeld double decomposition, into two three-dimensional Lie algebras. The associated three-dimensional groups, G and $\tilde{G}$ have generators denoted by T_a and $\tilde{T}^a$, where $a = 1, 2, 3$. It is convenient to split the index $a = (3, \alpha)$, with $\alpha = 1, 2$. Abelian subgroups are generated by T_3 and $\tilde{T}^3$, so that α takes values in the corresponding two-dimensional coset spaces. The non-vanishing structure constants of the algebras next to be considered are

$$\begin{aligned} IX &: f_{\alpha\beta}{}^{3} = f_{3\alpha}{}^{\beta} = \epsilon_{\alpha\beta}\,, & V &: f_{3\alpha}{}^{\beta} = \delta_{\alpha\beta}\,, \\ VII_0 &: f_{3\alpha}{}^{\beta} = \epsilon_{\alpha\beta}\,, & II &: f_{\alpha\beta}{}^{3} = \epsilon_{\alpha\beta}\,, \end{aligned} \tag{5.1}$$

where we have labeled them according to the standard Bianchi classification for three dimensional algebras. Only four combinations of the above correspond to six-dim Drinfeld doubles, namely $(IX, V), (VII_0, II), (II, V), (V, VII_0)$, in a $(G, \tilde{G})$ notation. The renormalization group equations for the first pair has been constructed in [10] whereas a classification of all six-dimensional doubles based on the Bianchi classification of three-dimensional algebras can be found in [23] (see also [24] for further related issues).

Since $f_{\alpha\beta}{}^{\gamma} = \tilde{f}^{\alpha\beta}{}_{\gamma} = 0$, we may according to the results of subsection 4.1 take consistently the form of the matrix M^{ab} as block diagonal, namely

$$(M^{ab}) = \begin{pmatrix} 1/G & 0 \\ 0 & K^{\alpha\beta} \end{pmatrix}\,, \qquad (K^{\alpha\beta}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,. \tag{5.2}$$

Taking the coset limit (4.5), whenever possible, corresponds to $G \to 0$.

The case of (IX, V). This case was examined in detail in [14]. The beta-function equations for the general matrix in (5.2), are quiet lengthy and they will not be presented here. However, for $K^{\alpha\beta}$ satisfying (4.9), the expressions become much simpler. Setting $A = D = a$, $B = -C = b - 1$ and $G = (1 + g)/a$ we find

$$\begin{aligned} \frac{da}{dt} &= \frac{1 + a^2 - b^2}{2a^2}((g-1)a^2 + (g+1)(b^2-1))\,, \\ \frac{db}{dt} &= \frac{b}{a}(a^2(g-1) + (g+1)(b^2-1))\,, \\ \frac{dg}{dt} &= \frac{1+g}{a}(g(1+a^2) + (g+2)b^2)\,, \end{aligned} \tag{5.3}$$

which are precisely the expressions in eq. (4.4) in [14]. The coset limit is $g \to -1$.

The case of (VII_0, II). It turns out that in this case the coset construction is not possible, i.e. (4.9) has no solution. To simplify the renormalization group flow, we present just a consistent truncation of the beta-function equations namely, along $A = D$ and $B = -C$

$$\frac{dA}{dt} = -\frac{1}{2AG}\,, \qquad \frac{dB}{dt} = 0\,, \qquad \frac{dG}{dt} = -\frac{1}{2A^2}\,. \tag{5.4}$$

The case of (II, V). In this case there is no constraint on the coset construction, i.e. no constraint on $K^{\alpha\beta}$. We present just a consistent truncation of these equations namely, along $A = D, B = -C$

$$\begin{aligned}\frac{dA}{dt} &= \frac{(A^4 - B^2(2+B)^2)}{2A}G\,,\\ \frac{dB}{dt} &= (1+B)(A^2 + B(2+B))G\,,\\ \frac{dG}{dt} &= \frac{(A^2+B^2)(A^2+(2+B)^2)}{2A^2}G^2\ .\end{aligned} \tag{5.5}$$

The case of (V, VII_0). A consistent truncation of the beta-function equations is along $A = D$ and $B = -C$

$$\frac{dA}{dt} = -\frac{2B(B+AG)}{AG}\,, \qquad \frac{dB}{dt} = 2\left(A + \frac{B}{G}\right)\,, \qquad \frac{dG}{dt} = -2 - \frac{2B^2}{A^2}\ . \tag{5.6}$$

5.2 Flow in SU(3) and the coset SU(3)/SU(2)

We use the structure constants in the Gell-Mann basis for SU(3) (see, for instance, eq. (5.2) of the second of [22])

$$f_{12}{}^3 = 2\,, \quad f_{14}{}^7 = -f_{15}{}^6 = f_{24}{}^6 = f_{25}{}^7 = f_{34}{}^5 = -f_{36}{}^7 = 1\,, \quad f_{45}{}^8 = f_{67}{}^8 = \sqrt{3}\,, \tag{5.7}$$

where the rest are obtained by antisymmetrization and pick up as an SU(2) subgroup the one generated by T_i, $i = 1, 2, 3$. Then in is easy to check that the most general invariant matrix K is

$$K = \begin{pmatrix} A & C & D & Z & 0\\ -C & A & Z & -D & 0\\ -D & -Z & A & C & 0\\ -Z & D & -C & A & 0\\ 0 & 0 & 0 & 0 & B \end{pmatrix}\,, \tag{5.8}$$

which has diagonal symmetric part as well as an antisymmetric one.

For the renormalization group flow in the full SU(3) model we take for the matrix M the form (4.1) with $M^{i\alpha} = M^{\alpha i} = 0$ and $H = \frac{1}{G}\,\mathbb{I}_{3\times 3}$. We find that the system (3.11) leads to a consistent system and in particular the r.h.s. of (4.2) is zero even though the coset SU(3)/SU(2) is not a symmetric space. We present the equations in the particular case of zero antisymmetric part, i.e. when $C = D = Z = 0$, which is a consistent truncation. We obtain

$$\frac{dA}{dt} = \frac{3A^2}{2}\frac{A(BG+1) - 4B}{B}\,, \qquad \frac{dB}{dt} = -3A^2\,, \qquad \frac{dG}{dt} = G^2A^2 + 2\ . \tag{5.9}$$

For the coset SU(3)/SU(2) the flow equations are the first two in the limit $G \to 0$.

Similar results can be found for the models SO(4)/SO(3) and Sp(4)/(SU(2) × U(1)) cases as well, but we will not present the details.

JHEP06(2009)082

6 Conclusions and future directions

Two-dimensional σ-models related classically by Poisson-Lie T-duality are one-loop renormalizable in the usual field theoretical sense in which all infinities can be absorbed into the coupling constants, as it was proven generally in [13]. In the present paper, using results of the latter work we proved the quantum equivalence under one-loop renormalization group flow of general σ-models related by Poisson-Lie T-duality, by explicitly demonstrating that the systems of beta-function equations obtained in each model separately are in fact equivalent. Hence, in that respect, [13] and our present work extend in full generality previous results based on low dimensional examples [10, 14]. In addition, we have studied general properties of the flows, as well as the associated generalized coset models in this context. Finally, we provided explicit examples based mainly on three-dimensional algebras in the Bianchi classification.

It would be very interesting to formulate the renormalization group flow in a duality-invariant way. For this an appropriate starting point should be the duality-invariant action formulation of Poisson-Lie T-duality in [25]. Since this action has twice as many fields as (2.1) and (2.2) and in addition in lacks two-dimensional Lorentz invariance, the corresponding one-loop counter-terms should be derived as a necessary first step.

In trying to explicitly solve the system of the beta-functions for low dimensional models it helps to know the renormalization group invariants. This can be possibly worked out example by example, for instance for the flows of the previous section. For the system (5.3) such an invariant was found in [14] and indeed helped in reducing it into an single, first order non-linear differential equation. However, for general considerations it would be interesting to have the forms of some if not of all of such invariants and classify them using the underlying group theoretic structure of Poisson-Lie T-duality.

As indicated above, in all known examples canonically equivalent classical σ-models are also equivalent under one-loop renormalization group flow in the sense already explained. It is interesting to search and provide a general proof of that statement.

Acknowledgments

K. Siampos acknowledges support by the Greek State Scholarship Foundation (IKY).

References

[1] K. Kikkawa and M. Yamasaki, *Casimir effects in superstring theories*, *Phys. Lett.* **B 149** (1984) 357 [SPIRES];
N. Sakai and I. Senda, *Vacuum energies of string compactified on torus*, *Prog. Theor. Phys.* **75** (1986) 692 [*Erratum ibid.* **77** (1987) 773] [SPIRES];
T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59 [SPIRES].

[2] B.E. Fridling and A. Jevicki, *Dual representations and ultraviolet divergences in nonlinear σ-models*, *Phys. Lett.* **B 134** (1984) 70 [SPIRES];

JHEP06(2009)082

E.S. Fradkin and A.A. Tseytlin, *Quantum equivalence of dual field theories*, *Ann. Phys.* **162** (1985) 31 [SPIRES];
X.C. de la Ossa and F. Quevedo, *Duality symmetries from non-abelian isometries in string theory*, *Nucl. Phys.* **B 403** (1993) 377 [hep-th/9210021] [SPIRES].

[3] C. Klimčík and P. Ševera, *Dual non-Abelian duality and the Drinfeld double*, *Phys. Lett.* **B 351** (1995) 455 [hep-th/9502122] [SPIRES];
C. Klimčík, *Poisson-Lie T-duality*, *Nucl. Phys. Proc. Suppl.* **46** (1996) 116 [hep-th/9509095] [SPIRES].

[4] K. Sfetsos, *Poisson-Lie T-duality and supersymmetry*, *Nucl. Phys. Proc. Suppl.* **56B** (1997) 302 [hep-th/9611199] [SPIRES].

[5] K. Sfetsos, *Canonical equivalence of non-isometric σ-models and Poisson-Lie T-duality*, *Nucl. Phys.* **B 517** (1998) 549 [hep-th/9710163] [SPIRES].

[6] T. Curtright and C.K. Zachos, *Currents, charges and canonical structure of pseudodual chiral models*, *Phys. Rev.* **D 49** (1994) 5408 [hep-th/9401006] [SPIRES].

[7] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *A canonical approach to duality transformations*, *Phys. Lett.* **B 336** (1994) 183 [hep-th/9406206] [SPIRES].

[8] C. Klimčík and P. Ševera, *Open strings and D-branes in WZNW models*, *Nucl. Phys.* **B 488** (1997) 653 [hep-th/9609112] [SPIRES].

[9] C. Albertsson and R.A. Reid-Edwards, *Worldsheet boundary conditions in Poisson-Lie T-duality*, *JHEP* **03** (2007) 004 [hep-th/0606024] [SPIRES].

[10] K. Sfetsos, *Poisson-Lie T-duality beyond the classical level and the renormalization group*, *Phys. Lett.* **B 432** (1998) 365 [hep-th/9803019] [SPIRES].

[11] V.E. Zakharov and A.V. Mikhailov, *Relativistically invariant two-dimensional models in field theory integrable by the inverse problem technique* (in russian), *Sov. Phys. JETP* **47** (1978) 1017 [*Zh. Eksp. Teor. Fiz.* **74** (1978) 1953] [SPIRES].

[12] C.R. Nappi, *Some properties of an analog of the nonlinear σ-model*, *Phys. Rev.* **D 21** (1980) 418 [SPIRES].

[13] G. Valent, C. Klimčík and R. Squellari, *One loop renormalizability of the Poisson-Lie σ-models*, arXiv:0902.1459 [SPIRES].

[14] K. Sfetsos, *Duality-invariant class of two-dimensional field theories*, *Nucl. Phys.* **B 561** (1999) 316 [hep-th/9904188] [SPIRES].

[15] R. Von Unge, *Poisson-Lie T-plurality*, *JHEP* **07** (2002) 014 [hep-th/0205245] [SPIRES].

[16] L. Hlavaty and L. Snobl, *Poisson-Lie T-plurality as canonical transformation*, *Nucl. Phys.* **B 768** (2007) 209 [hep-th/0608133] [SPIRES].

[17] L. Hlavaty and L. Snobl, *Description of D-branes invariant under the Poisson-Lie T-plurality*, *JHEP* **07** (2008) 122 [arXiv:0806.0963] [SPIRES].

[18] D. Friedan, *Nonlinear models in two epsilon dimensions*, *Phys. Rev. Lett.* **45** (1980) 1057 [SPIRES];
L. Álvarez-Gaumé, D.Z. Freedman and S. Mukhi, *The background field method and the ultraviolet structure of the supersymmetric nonlinear σ-model*, *Ann. Phys.* **134** (1981) 85 [SPIRES];
E. Braaten, T.L. Curtright and C.K. Zachos, *Torsion and geometrostasis in nonlinear σ-models*, *Nucl. Phys.* **B 260** (1985) 630 [SPIRES].

JHEP06(2009)082

[19] B.E. Fridling and A.E.M. van de Ven, *Renormalization of generalized two-dimensional nonlinear σ-models*, *Nucl. Phys.* **B 268** (1986) 719 [SPIRES].

[20] H. Osborn, *General bosonic σ-models and string effective actions*, *Ann. Phys.* **200** (1990) 1 [SPIRES].

[21] J. Balog, P. Forgacs, Z. Horvath and L. Palla, *Perturbative quantum (in)equivalence of dual σ-models in two-dimensions*, *Nucl. Phys. Proc. Suppl.* **49** (1996) 16 [hep-th/9601091] [SPIRES]; *Quantum corrections of Abelian duality transformations*, *Phys. Lett.* **B 388** (1996) 121 [hep-th/9606187] [SPIRES];
L.K. Balazs et al., *Quantum equivalence of σ-models related by non-Abelian duality transformations*, *Phys. Rev.* **D 57** (1998) 3585 [hep-th/9704137] [SPIRES].

[22] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Acedemic, New York U.S.A. (1978);
F. Mueller-Hoissen and R. Stuckl, *Coset spaces and ten-dimensional unified theories*, *Class. Quant. Grav.* **5** (1988) 27 [SPIRES].

[23] M.A. Jafarizadeh and A. Rezaei-Aghdam, *Poisson-Lie T-duality and Bianchi type algebras*, *Phys. Lett.* **B 458** (1999) 477 [hep-th/9903152] [SPIRES].

[24] L. Snobl and L. Hlavaty, *Classification of 6-dimensional real Drinfeld doubles*, *Int. J. Mod. Phys.* **A 17** (2002) 4043 [math/0202210] [SPIRES].

[25] C. Klimčík and P. Ševera, *Poisson-Lie T-duality and loop groups of Drinfeld doubles*, *Phys. Lett.* **B 372** (1996) 65 [hep-th/9512040] [SPIRES].

JHEP06(2009)082